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NOTE

Extending Large Sets of t-Designs

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recursive methods to construct large sets of t-designs. As an application, we construct infinite families of large sets of t-designs for all t. In particular, we show that if $v=2^{t-3}m-2$, $k=2^{t-3}-1$, and $t, m \geqslant 2$, then a $LS(\binom{v-t}{k-t}/2; t, k, v)$ exists.
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1. INTRODUCTION

Let $v, k, t,$ and λ be four positive integers such that $v \geqslant k \geqslant t > 0$. We denote the set of i-subsets of a set X by $P_i(X)$.

A t-design $S(\lambda; t, k, v)$ is a pair $(X, \mathscr{B})$ in which X is a finite set with cardinality v and $\mathscr{B}$ is a collection of elements of $P_k(X)$ such that every element of $P_t(X)$ appears exactly λ times in $\mathscr{B}$.

A large set of disjoint $S(\lambda; t, k, v)$ designs, denoted by $LS(\lambda; t, k, v)$, is a partition of the k-subsets of a v-set into $S(\lambda; t, k, v)$ designs. Obviously, if a $LS(\lambda; t, k, v)$ exists, then $\lambda = \binom{v-t}{k-t}/n$ for some n. For the sake of simplicity, we write $LS(1/n; t, k, v)$ instead of $LS(\binom{v-t}{k-t}/n; t, k, v)$.

A well known necessary condition for the existence of a $LS(1/n; t, k, v)$ is that $n \mid \binom{v-i}{k-i}$ for $i = 0, ..., t$. It is well known that in each of the following cases these conditions are also sufficient: (i) $t=1$, [2] (ii) $t=2, k=3$ and $v \neq 7$, [3,7-11,13-15] (iii) $t=2, k \leqslant 15$ and $n=2$, [1, 4] (iv) $t=3, k=4$ and $v \equiv 0 \pmod 3$, [13] (v) $t=6, k=7$ and $n=2$, [5, 6, 12] (vi) $v = nl(t)+t$ and $k=t+1$ in which $l(t)$ is defined recursively by $l(i) = l(i-1)$ $l.c.m.\{1, ..., i+1\}$ $l.c.m.\{\binom{i}{j} \mid j=1, ..., i\}$, $l(0)=1$ [12]. For $t>6$, designs of the last class together with their derived and residual designs are the only known designs.

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In this paper, we develop some recursive methods to construct large sets of t-designs. As an application, we show that if $v=2^{t-3}m-2$, $k=2^{t-3}-1$, and $m \geqslant 2$, then a $LS(\binom{v-t}{k-t}/2; t, k, v)$ exists.

2. BACKGROUND

Let X_1 and X_2 be two finite sets and k_1 and k_2 be two positive integers. Then for $\mathscr{B}_1 \subseteq P_{k_1}(X_1)$ and $\mathscr{B}_2 \subseteq P_{k_2}(X_2)$, and $X_1 \cap X_2 = \varnothing$, we define

$$\mathscr{B}_1 * \mathscr{B}_2 = \{A_1 \cup A_2 \mid A_1 \in \mathscr{B}_1 \ \& \ A_2 \in \mathscr{B}_2\}.$$

Clearly $\mathscr{B}_1 * \mathscr{B}_2 \subseteq P_{k_1+k_2}(X_1 \cup X_2)$.

Let $\mathscr{B} \subseteq P_k(X)$, For each $T \in P_t(X)$ the number of occurences of T in the blocks of $\mathscr{B}$ will be denoted by $n(T; \mathscr{B})$. Clearly $n(T; \mathscr{B})=0$ whenever $t>k$.

Let X be a finite set, and let k and t be two positive integers such that $t<k$. Two subsets A and B of $P_k(X)$ are said to be t-wise equivalent if the number of the occurences of each $T \in P_t(X)$ in A and B are the same. In particular, A and B are 0-wise equivalent if and only if $|A|=|B|$. A subset of $P_k(X)$ is called (n, t)-partitionable if it has a partition into n disjoint t-wise equivalent subsets. The following lemmas are proved in [1].

LEMMA 2.1. (i) *If* $0 \leqslant i \leqslant t$, *then t-wise equivalence implies i-wise equivalence, and* (ii) *a disjoint union of* (n, t)*-partitionable sets is* (n, t)*-partitionable.*

LEMMA 2.2. *Let* X_1 *and* X_2 *be two disjoint sets, and let* t_1, t_2, k_1, *and* k_2 *be four positive integers such that* $0 \leqslant t_1 \leqslant k_1$ *and* $0 \leqslant t_2 \leqslant k_2$. *For* $i=1, 2$, *let* $\mathscr{B}_i \subseteq P_{k_i}(X_i)$, *and suppose that* $\mathscr{B}_1$ *is* (n, t_1)*-partitionable. Then*

(i) $\mathscr{B}_1 * \mathscr{B}_2$ *is* (n, t_1)*-partitionable,*

(ii) *if* $\mathscr{B}_2$ *is* (n, t_2)*-partitionable, then* $\mathscr{B}_1 * \mathscr{B}_2$ *is* (n, t_1+t_2+1)*-partitionable.*

LEMMA 2.3. *If a* $LS(1/n; t, k, v)$ *and a* $LS(1/n; t, k+1, v)$ *exist, then a* $LS(1/n; t, k+1, v+1)$ *also exists.*

3. MAIN RESULTS

Throughout this section, we assume p is a prime, u, k and n are positive integers such that $np \leqslant k < (n+1)p$ and $u > n$. Let $X=\{1, ..., up\}$ and

$A_i = \{(i-1)p+1, ..., ip\}$ for $1 \leqslant i \leqslant u$. Let $Y = \{A_1, ..., A_u\}$ and order Y by

$$A_i < A_j \qquad \text{if and only if} \quad i < j$$

We define a function ϕ from the power set of Y into the power set of X by

$$\phi(B) = \bigcup_{A_i \in B} A_i, \qquad \text{for} \quad B \subset Y.$$

The following lemma is immediate.

LEMMA 3.1. *If $\mathscr{B}_1$ and $\mathscr{B}_2$ are j-wise equivalent subsets of $P_m(Y)$, then $\phi(\mathscr{B}_1)$ and $\phi(\mathscr{B}_2)$ are j-wise equivalent subsets of $P_{mp}(X)$.*

Let $l, m, a_1, ..., a_l$ be nonnegative integers such that $l \geqslant 1$, $1 \leqslant a_i < p$ $(1 \leqslant i \leqslant l)$ and $k = mp + \sum_{i=1}^{l} a_i$. Let $B \in P_k(X)$ and $T = \{C_1, ..., C_l\} \in P_l(Y)$, $C_1 < \cdots < C_l$. Define

$$Supp(B) = \{A_i \mid A_i \cap B \neq \varnothing\},$$

$$f(B) = \{A_i \mid A_i \subset B\},$$

$$g(B) = Supp(B) \backslash f(B),$$

$$\Gamma(T, a_1, ..., a_l) = \{B \subset X \mid |B \cap C_i| = a_i \;\&\; g(B) = Supp(B) = T\},$$

$$\mathscr{F}_m(T, a_1, ..., a_l) = \{B \in P_k(X) \mid |B \cap C_i| = a_i \;\&\; g(B) = T\},$$

and let

$$\mathscr{P} = \left\{ \mathscr{F}_m(T, a_1, ..., a_l) \mid l \geqslant 1, T \in P_l(Y), 1 \leqslant a_i < p, \& \sum_{i=1}^{l} a_i = k - mp \right\}.$$

Clearly, we have

$$\bigcup \{A \mid A \in \mathscr{P}\} = \{B \in P_k(X) \mid g(B) \neq \varnothing\}.$$

Therefore we have the following lemma.

LEMMA 3.2. (i) *If $k \neq np$, then $\mathscr{P}$ is partition of $P_k(X)$, and* (ii) *if $k = np$, then $\mathscr{P}$ is a partition of $P_k(X) \backslash \phi(P_n(Y))$.*

LEMMA 3.3. *If $T \in P_l(Y)$, and $1 \leqslant a_i < p$ $(1 \leqslant i \leqslant l)$, then $\Gamma(T, a_1, ..., a_l)$ is $(p, l-1)$ partitionable.*

Proof. Let $T=\{C_1, ..., C_l\}$ with $C_1<\dots C_l$. Then

$$\Gamma(T, a_1, ..., a_l)=\Gamma(T\backslash\{C_l\}, a_1, ..., a_{l-1}) * P_{a_l}(C_l).$$

Since p is prime, $p \mid \binom{p}{a_l}$, and so $P_{a_l}(C_l)$ is $(p, 0)$-partitionable. Now, the assertion follows by induction on l.

LEMMA 3.4. *If $T\in P_l(Y)$, and $1\leqslant a_i<p$ $(1\leqslant i\leqslant l)$, then*

(i) *$\mathscr{F}_m(T, a_1, ..., a_l)$ is $(p, l-1)$ partitionable,*

(ii) *if a $LS(1/p;s, m, u-l)$ exists, then $\mathscr{F}_m(T, a_1, ..., a_l)$ is $(p, s+l)$-partitionable.*

Proof. It is easy to check that

$$\mathscr{F}_m(T, a_1, ..., a_l)=\phi(P_m(Y\backslash T)) * \Gamma(T, a_1, ..., a_l).$$

Now, the assertion is an immediate consequence of Lemmas 2.1, 3.1, and 3.3.

LEMMA 3.5. *If a $LS(1/p; t_1, k_1, v_1)$ exists, then for $0\leqslant i\leqslant t_1$ and $j\in\{k_1-i, ..., k_1\}$, a $LS(1/p; t_1-i, j, v_1-i)$ exist.*

Proof. Let $\{(X, \mathscr{B}_l)|1\leqslant l\leqslant p\}$ be a $LS(1/p; t_1, k_1, v_1)$, and let Y_1 and Y_2 be two subsets of X such that $Y_1\cap Y_2=\varnothing$, $|Y_1|=k_1-j$, and $|Y_2|=i+j-k_1$, and define

$$\mathscr{B}'_l=\{B\backslash Y_1 \mid Y_2\cap B=\varnothing \;\&\; Y_1\subset B\in\mathscr{B}_l\}.$$

Then $\{(X\backslash(Y_1\cup Y_2), \mathscr{B}'_l) \mid 1\leqslant l\leqslant p\}$ is a $LS(1/p; t_1-i, j, v_1-i)$. ∎

THEOREM 1. *If a $LS(1/p; t, n, u)$ exists, then a $LS(1/p; t, pn, pu)$ also exists.*

Proof. Let $k=np$ and $\mathscr{F}_m(T, a_1, ..., a_l)\in\mathscr{P}$. If $l\leqslant t$, then $(n-m)p=\sum_{i=1}^{l} a_i<lp$, so $m\geqslant n-l$. Therefore, by Lemmas 3.4, and 3.5 $\mathscr{F}_m(T, a_1, ..., a_l)$ is (p, t)-partitionable. By the assumption and Lemma 3.1 $\phi(P_n(Y))$ is also (p, t)-partitionable. Now, the assertion follows by Lemmas 2.1, 3.2 and 3.3.

THEOREM 2. *If a $LS(1/p; t, n, u-1)$ exists and $np<k<(n+1)p$, then a $LS(1/p; t+1, k, pu)$ also exist.*

Proof. Let $\mathscr{F}_m(T, a_1, ..., a_l)\in\mathscr{P}$. If $l\leqslant t+1$, then $(n-m)p=\sum_{i=1}^{l} a_i<lp$, so $m>n-l$. Therefore, by Lemmas 3.4, and 3.5 $\mathscr{F}_m(T, a_1, ..., a_l)$ is

$(p, t+1)$-partitionable. Now, the assertion follows from Lemmas 2.1, 3.2, and 3.3.

THEOREM 3. *If a* $LS(1/p; t, n, u-1)$ *exists and* $1 \leqslant j < i < p-1$, *then a* $LS(1/p; t+1, np+i, pu+j)$ *also exists.*

Proof. The assertion is an immediate consequence of Lemma 2.3 and Theorem 2. ∎

Now we apply Theorem 2 to find some infinite families of large sets of t-designs for all t.

THEOREM 4. *If* $t \geqslant 6$ *and* $m \geqslant 2$, *then a* $LS(1/2; t, 2^{t-3}-1, m2^{t-3}-2)$ *exists.*

Proof. It is well known that if $m \geqslant 2$, then a $LS(1/2; 6, 7, 8m-2)$ exists [5, 6, 12]. Now, the result follows by induction on t (and applying Theorem 2).

THEOREM 5. *Let p be any odd prime, and* $l, t, m, a_1, ..., a_{t-1}$ *be positive integers such that* $t, l, m \geqslant 1$ *and* $1 \leqslant a_i < p$. *Then a* $LS(1/p; t, \sum_{i=1}^{t} a_i p^{t-i} + mp^t, (l-1)\, p^{t+1} + \sum_{i=1}^{t} p^i)$ *exists.*

Proof. Trivially for $1 \leqslant m < p$ a $LS(1/p; 0, m, lp)$ exists. Now, the assertion follows by induction on t.

REFERENCES

1. S. Ajoodani-Namini and G. B. Khosrovshahi, More on halving complete designs, *Discrete Math.* **135** (1994), 29–37.
2. Z. Baranyai, On the factorizations of complete uniform hypergraph, *in* "Finite and Infinite Sets," Colloq. Math. Soc. Janos Bolyai, Vol. 10, pp. 91–108, North-Holland, Amesterdam, 1975.
3. J. Doyen, Construction of disjoint Steiner triple systems, *Proc. Amer. Math. Soc.* **32** (1972), 409–416.
4. A. Hartman, Halving the incomplete design, *Ann. of Discrete Math.* **34** (1987), 207–224.
5. G. B. Khosrovshahi and S. Ajoodani-Namini, Combining t-designs, *J. Combin. Theory Ser. A* **58** (1991), 26–34.
6. D. L. Kreher and S. P. Radsznowski, The existence of simple 6-(14, 7, 4) designs, *J. Combin. Theory Ser. A* **43** (1986), 237–243.
7. J. X. Lu, On large sets of disjoint Steiner triple systems, I,II,III, *J. Combin. Theory Ser. A* **34** (1983), 140–146, 147–155, 156–182.
8. J. X. Lu, On large sets of disjoint Steiner triple systems, IV, V, VI, *J. Combin. Theory Ser. A* **37** (1984), 136–163, 164–188, 189–192.
9. A. Rosa, A theorem on the maximum number of disjoint Steiner triple systems, *J. Combin. Theory Ser A* **18** (1975), 305–312.
10. Schreiber, Some balanced complete block designs, *Israel J. Math.* **8** (1974), 31–37.

11. L. Teirlinck, A completion of Lu's determination of the spectrum for large sets of Steiner triple systems, *J. Combin. Theory Ser. A* **57** (1991), 302–305.
12. L. Teirlinck, Locally trivial *t*-designs and *t*-designs without repeated blocks, *Discuss. Math.* **77** (1989), 345–356.
13. L. Teirlinck, On large sets of disjoint quadruple systems, *Ars Combin.* **17** (1984), 173–176.
14. L. Teirlinck, On the maximum number of disjoint Steiner triple systems, *Discrete Math.* **6** (1973), 299–300.
15. L. Teirlinck, On the maximum number of disjoint triple systems, *J. Geometry* **6** (1975), 93–96.